

Correlation Functions of the Critical Ashkin–Teller Model on a Torus

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The correlation functions of the critical Ashkin–Teller model on a torus are calculated using a free bosonic field formulation in the continuum limit. The results include in particular correlators of electromagnetic or twist operators. Various applications are discussed.

KEY WORDS: Correlation functions on a torus; twist operators.

1. INTRODUCTION

Much work has been devoted recently to the study of the partition functions of 2D critical systems on a torus.⁽¹⁾ For instance, the constraint of modular invariance has allowed a systematic classification^(2–5) of all minimal⁽⁶⁾ conformal theories with central charge $c < 1$, determining their full operator content. Also, free field constructions involving various types of topological defects have been proposed,^(7–10) establishing links with the standard Coulomb gas mappings of statistical mechanics⁽¹¹⁾ as well as with recent developments in string theory.⁽¹²⁾

The correlation functions on the torus are also of interest, in particular because they provide a systematic route to the study of deviations from criticality.⁽¹³⁾ Their construction appears, however, technically difficult, and only the Ising model has been considered.⁽¹⁴⁾ My purpose in this work is to extend the analysis of Ref. 14 to the whole critical line of the Ashkin–Teller (AT) model⁽¹⁵⁾ including, in particular, the Kosterlitz–Thouless, the Z_4 ,⁽¹⁶⁾ and the four-state Potts model points. The corresponding problem on the plane has been considered in Ref. 17. It turns out that some of the present results are also useful from the point of view of string theory on orbifolds.⁽¹⁸⁾

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The paper is organized as follows. Section 2 recalls the free field mapping^(11,19) of the AT model, which will be the central instrument of the present approach. I discuss the way physical operators translate⁽¹⁹⁾ into “electromagnetic” or “twist” fields and rederive the partition function^(9,10) on the torus. Section 3 calculates the n -point correlation functions for electromagnetic operators, and illustrates the results with various examples, such as the underlying Luttinger model fermions⁽²⁰⁾ or the energy. Section 4 considers the $2n$ twist-functions with application to order parameter correlators.⁽¹⁷⁾ Section 5 discusses the compatibility of the results with those derived by other approaches using covering Riemann surfaces.⁽²¹⁾ Section 6 contains a few final comments.

2. FREE FIELD FORMULATION OF THE ASHKIN–TELLER MODEL

1. The AT model⁽¹⁵⁾ consists of two Ising models coupled by a four spin interaction, with action

$$\mathcal{A} = - \sum_{\langle jk \rangle} K_2(S_j S_k + t_j t_k) + K_4 S_j S_k t_j t_k \tag{2.1}$$

where $\langle jk \rangle$ denotes nearest neighbors of the square lattice \mathcal{L} , and $S, t = \pm 1$. As shown by various authors,⁽¹¹⁾ model (2.1) at criticality can be reformulated as a solid-on-solid (SOS) surface model, and then mapped onto a free field. First it is convenient to rewrite (2.1) (using the invariance of the partition function Z^{AT} under $t \rightarrow St'$) as

$$\mathcal{A} = - \sum_{\langle jk \rangle} K_2 S_j S_k (1 + t'_j t'_k) + K_4 t'_j t'_k \tag{2.2}$$

and Z^{AT} reads

$$\begin{aligned} Z^{\text{AT}} = & \sum_{\{S, t'\}} \prod_{\langle jk \rangle} [\exp(K_4 t'_j t'_k)] \text{ch}(K_2 + K_2 t'_j t'_k) \\ & \times [1 + S_j S_k \text{th}(K_2 + K_2 t'_j t'_k)] \end{aligned} \tag{2.3}$$

The $\{t'\}$ configuration can be represented by putting bonds on the dual lattice \mathcal{D} , which separate two sites with opposite t' , as in an Ising low-temperature expansion. The product of square brackets can then be expanded as in an Ising high-temperature expansion with a bond on \mathcal{L} each time the $S_j S_k$ term is taken. Summation over $\{S, t'\}$ gives

$$Z^{\text{AT}} = 2^{-\mathcal{N}} [\exp(2\mathcal{N}K_4)] (\text{ch } 2K_2)^{2\mathcal{N}} \sum_{\text{graphs}} (\text{th } 2K_2)^l \left(\frac{\exp(-2K_4)}{\text{ch } 2K_2} \right)^d \tag{2.4}$$

where \mathcal{N} is the total number of sites. The graphs are formed by polygons on \mathcal{L} and \mathcal{D} with an even number of bonds attached to any point, the total numbers of bonds on each lattice being respectively l and d . If a given bond is present on \mathcal{D} , the product of the two corresponding spins is -1 and the th in (2.3) is zero; thus, the polygons on \mathcal{L} and \mathcal{D} do not intersect (Fig. 1). The model presents a critical line given by the self-duality condition $\exp(-2K_4) = \text{sh } 2K_2$, which terminates at $\text{coth } 2K_2 = 2$.

The graphs of (2.4) can alternatively be represented by six-vertex configurations (Fig. 2) on the surrounding lattice \mathcal{S} , here another square lattice the vertices of which are the midpoints of the edges of \mathcal{L} . A bond on \mathcal{L} or \mathcal{D} is associated to a vertex of type 1, ..., 4, such that arrows are reflected by it, with a net nonzero polarization. Edges with no bond are associated to vertices of type 5, 6. Once a possible vertex is chosen, the whole correspondence follows by induction. A given configuration of bonds is thus associated with a configuration of the six-vertex model (defined up to a reversal of all arrows) and vice versa. Along the critical line this six-vertex model becomes in fact an F -model with Boltzmann weights

$$W_1 = \dots = W_4 = 1, \quad W_5 = W_6 = \text{coth } 2K_2 \tag{2.5}$$

Now the F -model can be transformed into an SOS model by introducing height variables φ on the faces of \mathcal{S} , such that two neighbouring φ differ by $\pm \pi/2$, the highest being on the left of each arrow. It is finally argued that this SOS model renormalizes⁽¹¹⁾ onto a Gaussian model with the free field action

$$\mathcal{A} = \frac{g}{4\pi} \int |\nabla\varphi|^2 d^2x \tag{2.6}$$

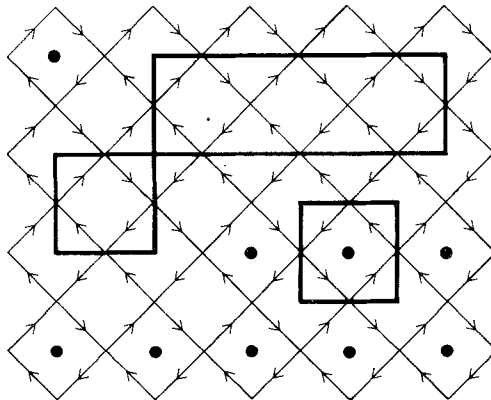


Fig. 1. A graph in the expansion (2.4) involving a polygon on \mathcal{L} (whose sites are indicated by dotted points) and a polygon on \mathcal{D} . The corresponding six-vertex configuration is indicated.

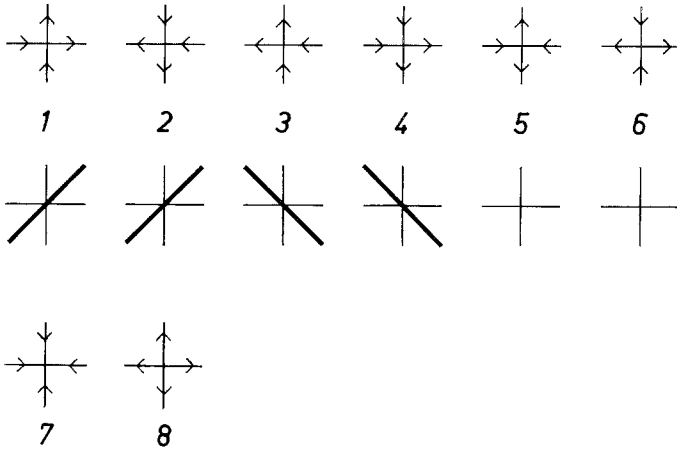


Fig. 2. Arrow configurations in the six-vertex model. Vertices of type 1,..., 4 are associated to bonds on \mathcal{L} or \mathcal{D} , vertices of type 5, 6 to edges with no bonds. Additional vertices of type 7, 8 correspond to introducing magnetic operators in the surface language.

Many operators in (2.1) translate then in the Gaussian language into combinations of spin wave (“electric”) operators O_E , i.e., exponentials of the field $e^{iE\varphi}$, and vortex (“magnetic”) operators O_M , which create a branch point with amplitude discontinuity of $2\pi M$ for the field φ (in the following we call E and M , respectively, electric and magnetic charges). Their dimensions and spin are⁽¹¹⁾

$$\begin{aligned} X_{EM} &= E^2/2g + gM^2/2 \\ S_{EM} &= EM \end{aligned} \tag{2.7}$$

Of particular interest is the operator $O_{0,\pm 1}$, which describes the introduction of a vertex of type 7, 8 (Fig. 2). From the solution of the eight-vertex model, one knows the singularity of the free energy in the neighborhood of $|W_7| = 0$,

$$f_s \sim |W_7|^{2/y}, \quad y = (4/\pi) \cos^{-1}(W_5/2) = 2 - x \tag{2.8}$$

from which one deduces via (2.5) and (2.7)

$$g = \frac{8}{\pi} \sin^{-1} \left(\frac{\coth 2K_2}{2} \right) \tag{2.9}$$

In a similar way the energy is known⁽¹⁹⁾ to be $\varepsilon \leftrightarrow O_{2,0}$ ($X_\tau = 2/g$) and the polarization operator ($P = St$) $P \leftrightarrow O_{1,0}$. The special points of the AT

line can then be identified⁽⁹⁾: $g = 1$ corresponds to the Kosterlitz–Thouless point, $g = 2$ to the decoupling (Ising) point, $g = 3$ to the Z_4 model of Ref. 16, and $g = 4$ the four-state Potts model.

The identification of the spin operator is of a different nature. It has been conjectured⁽²²⁾ for a long time that the magnetic exponent X_H remains constant along the critical line with the Ising value $X_H = 1/8$. Several authors have noticed^(9,23) that this feature is characteristic of the twist operator σ , which create a branch point singularity with change of sign of the field φ . The correspondence $S \leftrightarrow \sigma$ can in fact be established using the above model transformation, as shown in a slightly different way in Ref. 10. The two-spin correlation function $\langle S_j S_k \rangle$ is represented in the graph expansion (2.4) by adding a line on \mathcal{L} connecting j to k . Now, the six-vertex representation cannot be consistently achieved. Indeed, suppose we follow a curve encircling j or k ; then, defining recurrently the arrow configurations, one comes back to reversed orientations, since an odd number of lines on \mathcal{L} is crossed (Fig. 3). The new graphs thus correspond to antiperiodicity of the six-vertex model along a cut relating j to k . The heights being multiple of $\pi/2$, and a crossed line corresponding to a step of π , it is always possible to choose the origins such that this cut corresponds in turn to a change of sign of φ , establishing the desired result. This is also valid for $\langle t_j t_k \rangle$.

Higher order correlation functions are slightly more complicated, as we illustrate now for $\langle S_j S_k S_l S_m \rangle$. There is first a change of sign $\varphi \rightarrow -\varphi$

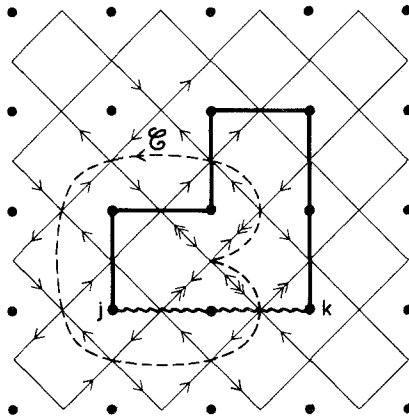


Fig. 3. For a graph in the expansion of $\langle S_j S_k \rangle$, the six-vertex representation cannot be defined in a coherent way. Following a closed path encircling j or k and defining recurrently the arrow configurations, one comes back to reversed orientations (double arrows on the figure). This is equivalent to antiperiodic conditions for the vertex model along a cut (wavy line) relating j to k , and also to antiperiodicity for the heights φ .

encircling any of the points, as was the case for $\langle S_j S_k \rangle$, resulting now in two cuts on the plane. In addition, the heights cannot be consistently defined on closed paths \mathcal{C} encircling two points, although the vertex configurations can. Depending on the number of polygons between the points, the \mathcal{C} and spin line configurations, the height varies in fact by an amount of $2m\pi$, $m \in \mathbb{Z}$, since the number of crossed bonds along \mathcal{C} is always even ($m = 1$ in Fig. 4). For the four-point function there are two independent contours, one possibility of which is represented Fig. 4b (for the $2n$ -point function, there would be $2n - 2$ such contours) and the correlation function is obtained by summing over “frustrations” m_1, m_2 ; see Section 4. We can also obtain mixed correlators such as $\langle S_j t_k S_l t_m \rangle$, which becomes in the variables of the graph expansion $\langle S_j S_k S_l S_m t'_k t'_m \rangle$. The $t'_k t'_m$ is negative if there is an even number of \mathcal{D} polygons between k and m , which translates into $m_1 = 2n + 1$ in terms of height variations, and the correlation function is obtained by summing over m_1, m_2 with an additional factor $(-)^{m_1}$. This construction appears rather similar to the one used for correlators in string theory on orbifolds and explains the similarity between the results of Refs. 7 and 18.

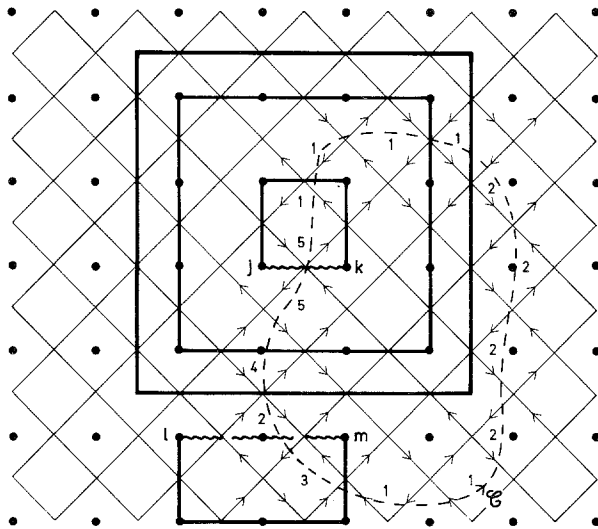
An alternative field-theoretic description of the AT model is provided by the Luttinger (massless Thirring) model, i.e., the interacting complex fermion theory with action

$$\mathcal{A} = \int (\psi \partial_z \psi^+ + \tilde{\psi} \partial_z \tilde{\psi}^+ + \Gamma \psi \psi^+ \tilde{\psi} \tilde{\psi}^+) d^2x \tag{2.10}$$

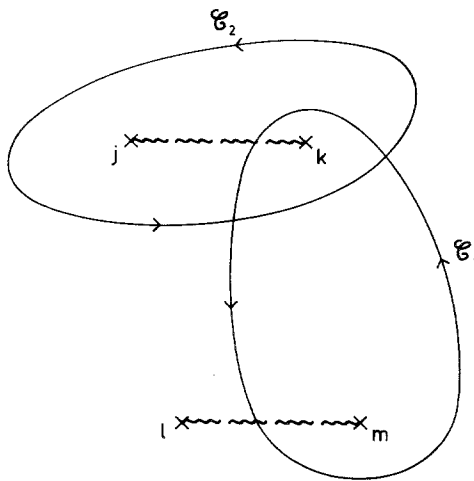
The relations between the two formulations have been studied in detail.⁽²⁰⁾ Recall simply that $\Gamma = (2 - g)/(2 + g)$ and the equivalences $\psi \leftrightarrow O_{1,1/2}$; $\psi^+ \leftrightarrow O_{-1,-1/2}$ ($\tilde{\psi} \leftrightarrow O_{-1,1/2}$; $\tilde{\psi}^+ \leftrightarrow O_{1,-1/2}$).

At the value $g = 3$, the AT model reduces to the Z_4 model of Ref. 16. The corresponding quarter-integer-spin parafermions χ, χ^+ are identified⁽⁹⁾ as $\chi \leftrightarrow O_{3/2,1/2}$; $\chi^+ \leftrightarrow O_{-3/2,-1/2}$ ($\tilde{\chi} \leftrightarrow O_{-3/2,1/2}$; $\tilde{\chi}^+ \leftrightarrow O_{3/2,-1/2}$).

2. So far we have discussed the AT model on an infinite plane. In a finite geometry, the boundary conditions generate various constraints on the successive transformations described above, resulting in a modified free field theory as discussed in detail in Ref. 10. The first point is that an asymmetry between \mathcal{L} and \mathcal{D} appears on a torus. Indeed, since a \mathcal{D} line corresponds to a t spin flip, any closed path has to cross an even number of \mathcal{D} bonds, while the number of \mathcal{L} crossed bonds can be arbitrary. If these numbers are both even, the reformulation as an F -model is still possible. Now, since height variables ϕ are associated *locally* to a vertex configuration, they cannot be defined in a consistent way; describing a noncontractible loop around the torus leads^(7,8) to a variation of height $\delta\phi$ which,



(a)



(b)

Fig. 4. A new feature appears in the construction of higher spin correlators, as illustrated here in the case of $\langle S_j S_k S_l S_m \rangle$. In addition to the cuts with change of sign of φ , the height cannot be consistently defined on curves encircling two points, although the six-vertex configurations can: there is indeed a variation of φ multiple of 2π : $\delta\varphi = +2\pi$ for the contour \mathcal{C} on part (a). The four-point spin function is obtained by summing over indices $m_1, m_2 \in \mathbb{Z}$ characterizing these frustrations for a set of two independent contours, an example of which is given in part (b).

due to the condition of an even number of crossed bonds, is a multiple of 2π . We describe the torus by two periods ω, ω' . Then, for a variation $\delta\varphi = 2\pi m, \delta'\varphi = 2\pi m'$ along the two generators, the corresponding continuum limit is the frustrated partition function introduced in Ref. 7,

$$Z_{mm'} = \int_{\substack{\delta\varphi = 2\pi m \\ \delta'\varphi = 2\pi m'}} [D\varphi] e^{-\mathcal{A}} \tag{2.11}$$

This is easily evaluated writing $\varphi \rightarrow \varphi + \varphi_{cl}$, where φ is now periodic and the ‘‘classical’’ part (such that $\Delta\varphi_{cl} = 0$) is given by

$$\varphi_{cl} = 2\pi \operatorname{Im} \left(\frac{m' - m\bar{\tau}}{\tau_1} z \right) \tag{2.12}$$

Then $Z_{mm'}$ factorizes as $Z_{mm'} = Z_0 \exp(-\mathcal{A}_{cl})$, where Z_0 is the ‘‘quantum’’ contribution, i.e., the partition function of the periodic free field⁽⁷⁾

$$Z_0 = \left(\frac{g}{\tau_1} \right)^{1/2} \frac{1}{|\eta(q)|^2} \tag{2.13}$$

and the classical action reads

$$\mathcal{A}_{cl} = \frac{g}{4\pi} \int_T |\nabla\varphi_{cl}|^2 d^2x = \pi g \frac{|m' - m\tau|^2}{\tau_1} \tag{2.14}$$

Thus,

$$Z_{mm'} = \left(\frac{g}{\tau_1} \right)^{1/2} \frac{1}{|\eta(q)|^2} \exp \left(-\pi g \frac{|m' - m\tau|^2}{\tau_1} \right) \tag{2.15}$$

In these expressions we have used the modular ratio $\tau = \omega'/\omega = \tau_R + i\tau_I$, and η is the Dedekind function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2i\pi\tau}$$

Summing over these ‘‘soliton’’ sectors indexed by m, m' gives the Coulomb partition function,^(7,9)

$$Z_C(g) = \sum_{mm' \in \mathbb{Z}} Z_{mm'} = \frac{1}{|\eta|^2} \sum_{em \in \mathbb{Z}} q^{h_{em}} \bar{q}^{\bar{h}_{em}} \tag{2.16}$$

(the last equality is obtained by a Poisson transformation on m'). The $q \rightarrow 0$ behavior $Z \sim (q\bar{q})^{-c/24}$ gives the central charge $c = 1$, and the conformal weights⁽⁶⁾ ($x = h + \bar{h}, s = h - \bar{h}$) are derived from (2.7),

$$h_{em} = \frac{1}{4} \left(\frac{e}{\sqrt{g}} + m\sqrt{g} \right)^2, \quad \bar{h}_{em} = \frac{1}{4} \left(\frac{-e}{\sqrt{g}} + m\sqrt{g} \right)^2 \tag{2.17}$$

Formula (2.16) appears also in the context of string theory as the partition function of a free field compactified on a circle. In the usual conventions,⁽¹²⁾ where the action is

$$\mathcal{A} = \frac{1}{2\pi} \int \partial_z \varphi \partial_{\bar{z}} \varphi$$

and φ is defined modulo $2\pi R$, the correspondence is given by $R = (2g)^{1/2}$.

To construct the partition function of the AT model, one must add the contribution of graphs where an odd number of \mathcal{D} bonds is crossed. By the same arguments leading to the identification of the spin as a twist operator, this condition translates with a correct choice of the height origins into antiperiodic boundary conditions for the field φ . Defining for $(\alpha\beta) \neq (00)$

$$Z_{(\alpha\beta)} = \int_{\substack{\varphi(z+1) = e^{2i\pi\alpha}\varphi(z) \\ \varphi(z+\tau) = e^{2i\pi\beta}\varphi(z)}} [D\varphi] e^{-\mathcal{A}} \tag{2.18}$$

which have been calculated in Ref. 24,

$$\begin{aligned} Z_{(1/2,0)} &= |\eta/\theta_4(0)| \\ Z_{(0,1/2)} &= |\eta/\theta_2(0)| \\ Z_{(1/2,1/2)} &= |\eta/\theta_3(0)| \end{aligned} \tag{2.19}$$

(here θ_v denotes the Jacobi theta function of argument $\tau^{(25)}$) one gets finally^(9,10)

$$Z^{\text{AT}} = \frac{1}{2} Z_{\text{C}}(g) + \sum_{v=2}^4 \left| \frac{\eta}{\theta_v(0)} \right| \tag{2.20}$$

The sum over the twisted sectors is also modular invariant. The relative normalization between the two contributions is not directly obtained by this approach, since there is a zero mode subtraction in the Coulombic sector only. It can be fixed by requiring the identity (spin) operators to be non (twice) degenerate. At the Ising decoupling point $g = 2$ one checks that

$$Z^{\text{AT}}(g = 2) = (Z^{\text{I}})^2 \tag{2.21}$$

where

$$Z^{\text{I}} = \frac{1}{2} \sum_{v=2}^4 \left| \frac{\theta_v(0)}{\eta} \right| \tag{2.22}$$

Formula (2.22) is the partition function of the Ising model, i.e., a free real fermion theory with action

$$\mathcal{A} = \int (\psi \partial_z \psi + \tilde{\psi} \partial_z \tilde{\psi}) d^2x \tag{2.23}$$

summed over four sectors of periodic (antiperiodic) boundary conditions for $(\psi, \tilde{\psi})$. The result (2.20) appears also in string theory as the partition function of a Z_2 -orbifold model.⁽⁹⁾

3. Finally we recall the existence for the model (2.6) of a duality transformation with the effect⁽¹⁹⁾

$$\begin{aligned} g &\rightarrow 1/g \\ O_{EM} &\rightarrow O_{ME} \\ \sigma &\rightarrow \sigma \end{aligned} \tag{2.24}$$

This leaves the twisted partition functions invariant, as well as the Coulombic one due to⁽⁷⁾

$$Z_C(g) = Z_C(1/g) \tag{2.25}$$

3. SPIN WAVE AND VORTEX OPERATORS CORRELATION FUNCTIONS

1. We first consider $\langle O_{EM}(z_1, \bar{z}_1) O_{-E,-M}(z_2, \bar{z}_2) \rangle$ in the doubly periodic sector. To take into account the discontinuity of $2\pi M$ on a cut relating 1 to 2 (Fig. 5a), we introduce the classical field

$$\varphi_{cl} = M \left\{ \text{Im} \log \left[\frac{\theta_1(z - z_1)}{\theta_1(z - z_2)} \right] - \frac{2\pi}{\tau_1} \text{Im} z \text{Re} z_{12} \right\} \tag{3.1}$$

It is doubly periodic, singular in z_1, z_2 , and satisfies $\Delta\varphi_{cl} = 0$ otherwise. Then, writing $\varphi \rightarrow \varphi + \varphi_{cl}$, the calculation of the functional integral gives, as in (2.12),

$$\begin{aligned} &\langle O_{EM}(1) O_{-E,-M}(2) \rangle_{pp} \\ &\propto \exp \left\{ iE[\varphi_{cl}(1) - \varphi_{cl}(2)] + E^2 \langle \varphi(1) \varphi(2) \rangle_{pp} - \frac{g}{4\pi} \int_T |\nabla\varphi_{cl}|^2 d^2x \right\} \end{aligned} \tag{3.2}$$

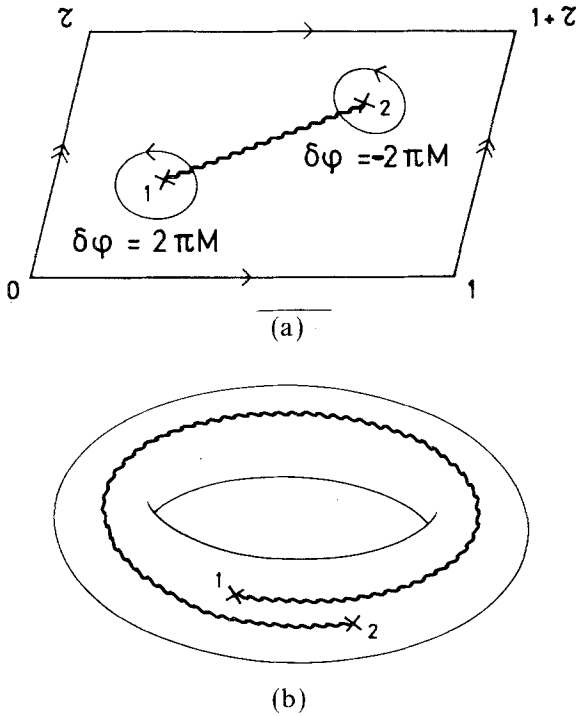


Fig. 5. (a) Magnetic operators create a cut with a discontinuity of $2\pi M$ for the field φ . (b) Shifting one of the points by $1, \tau, 1 + \tau$ is equivalent to wrapping a frustration line around the torus, and changes the soliton sector correspondingly.

where the propagator $\langle \varphi \varphi \rangle_{pp}$ has already been evaluated in Ref. 24,

$$\begin{aligned} \langle \varphi(1) \varphi(2) \rangle_{pp} &= -\frac{1}{g} \log \left\{ \left| \frac{\theta_1(z_{12})}{\theta_1'(0)} \right| \exp \left(-\pi \frac{\text{Im}^2 z_{12}}{\tau_1} \right) \right\} \\ &= -\frac{1}{2g} \log \Gamma(1, 2) \end{aligned} \tag{3.3}$$

To obtain the remaining integral, we introduce the other field

$$\phi_{cl} = M \left\{ \log \left| \frac{\theta_1(z - z_1)}{\theta_1(z - z_2)} \right| - \frac{2\pi}{\tau_1} \text{Re } z \text{ Re } z_{12} \right\} \tag{3.4}$$

We use Cauchy–Riemann relations to replace φ_{cl} by ϕ_{cl} , and perform then an integration by parts, which, due to

$$\Delta \phi_{cl} = 2\pi M [\delta^2(z - z_1) - \delta^2(z - z_2)] \tag{3.5}$$

(δ being understood as doubly periodic), gives finally

$$\begin{aligned} & \langle O_{EM}(1) O_{-E,-M}(2) \rangle_{pp} \\ &= \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{E^2/g + gM^2} \\ & \times \exp \left\{ \frac{\pi}{g\tau_1} (E \operatorname{Im} z_{12} - igM \operatorname{Re} z_{12})^2 - 2iEM \log \left[\frac{\theta_1(z_{12})}{\theta'_1(0)} \right] \right\} \end{aligned} \quad (3.6)$$

It will be useful to introduce

$$\delta_{EM} = \frac{1}{2} \left(\frac{E}{\sqrt{g}} + M \sqrt{g} \right), \quad \bar{\delta}_{EM} = \frac{1}{2} \left(-\frac{E}{\sqrt{g}} + M \sqrt{g} \right) \quad (3.7)$$

with which (3.6) reads

$$\begin{aligned} \langle O_{EM}(1) O_{-E,-M}(2) \rangle_{pp} &= \left[\frac{\theta'_1(0)}{\theta_1(z_{12})} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \\ & \times \exp \left[-\frac{\pi}{\tau_1} (\delta_{EM} z_{12} + \bar{\delta}_{EM} \bar{z}_{12})^2 \right] \end{aligned} \quad (3.8)$$

It is *not* the product of an analytic by an antianalytic function. If $M \neq 0$, it is not periodic, since shifting z_{12} by 1, τ , or $1 + \tau$ is equivalent to adding a new frustration line wrapping around the torus (Fig. 5b).

The corresponding expression in a soliton sector (m, m') is then easily obtained by adding (2.12) to the classical field (3.1), giving

$$\begin{aligned} & \langle O_{EM}(1) O_{-E,-M}(2) \rangle_{mm'} \\ &= \langle O_{EM}(1) O_{-E,-M}(2) \rangle_{pp} \\ & \times \exp \left[2i\pi E \operatorname{Im} \left(\frac{m' - m\bar{\tau}}{\tau_1} z_{12} \right) + 2\pi g M \operatorname{Re} \left(\frac{m' - m\bar{\tau}}{\tau_1} z_{12} \right) \right] \\ &= \left[\frac{\theta'_1(0)}{\theta_1(z_{12})} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \exp \left[-\frac{\pi}{\tau_1} (\delta_{EM} z_{12} + \bar{\delta}_{EM} \bar{z}_{12})^2 \right. \\ & \left. + 2\pi \sqrt{g} \left(\frac{m' - m\bar{\tau}}{\tau_1} \delta_{EM} z_{12} + \frac{m' - m\tau}{\tau_1} \bar{\delta}_{EM} \bar{z}_{12} \right) \right] \end{aligned} \quad (3.9)$$

and one can check that

$$\begin{aligned} & Z_{mm'} \langle O_{EM} O_{-E,-M} \rangle_{mm'}(z_{12} + 1) \\ &= Z_{m,m'-M} \exp[2i\pi E(m - M)] \langle O_{EM} O_{-E,-M} \rangle_{m,m'-M}(z_{12}) \end{aligned} \quad (3.10a)$$

$$\begin{aligned} & Z_{mm'} \langle O_{EM} O_{-E,-M} \rangle_{mm'}(z_{12} + \tau) \\ &= Z_{m+M,m'} \exp[2i\pi E(m' - M)] \langle O_{EM} O_{-E,-M} \rangle_{m+M,m'}(z_{12}) \end{aligned} \quad (3.10b)$$

The total correlation function in the Coulombic sector is

$$\langle O_{EM}(1) O_{-E,-M}(2) \rangle_C = \frac{1}{Z_C(g)} \sum_{mm' \in \mathbb{Z}} Z_{mm'} \langle O_{EM}(1) O_{-E,-M}(2) \rangle_{mm'} \tag{3.11}$$

Due to (3.10), it is periodic for E, M integers only.

2. We now consider a twisted sector, for instance, $(\alpha, \beta) = (1/2, 0)$. The classical field analogous to (3.1) then reads

$$\varphi_{cl} = M \operatorname{Im} \log \left[\frac{\theta_1((z-z_1)/2, \tau/2) \theta_2((z-z_2)/2, \tau/2)}{\theta_2((z-z_1)/2, \tau/2) \theta_1((z-z_2)/2, \tau/2)} \right] \tag{3.12}$$

satisfying $\varphi_{cl}(z+1) = -\varphi_{cl}(z)$, $\varphi_{cl}(z+\tau) = \varphi_{cl}(z)$, and the propagator⁽¹⁴⁾ is

$$\langle \varphi(1) \varphi(2) \rangle_{(1/2,0)} = -\frac{1}{2g} \log \left[\frac{\Gamma(z_{12}/2, \tau/2)}{\Gamma((z_{12}+1)/2, \tau/2)} \right] \tag{3.13}$$

Use of (3.2) gives in this case

$$\langle O_{EM}(1) O_{-E,-M}(2) \rangle_{(1/2,0)} = \left[\frac{\theta_2(z_{12}/2, \tau/2) \theta'_1(0, \tau/2)}{2\theta_1(z_{12}/2, \tau/2) \theta_2(0, \tau/2)} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \tag{3.14}$$

The results in the other sectors are obtained in a similar way and read

$$\begin{aligned} &\langle O_{EM}(1) O_{-E,-M}(2) \rangle_{(0,1/2)} \\ &= \left[\frac{\theta_4(z_{12}, 2\tau) \theta'_1(0, 2\tau)}{\theta_1(z_{12}, 2\tau) \theta_4(0, 2\tau)} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\langle O_{EM}(1) O_{-E-M}(2) \rangle_{(1/2,1/2)} \\ &= \left[\frac{\theta_2(z_{12}/2) \theta_4(z_{12}/2) \theta'_1(0) \theta_3(0)}{2\theta_1(z_{12}/2) \theta_3(z_{12}/2) \theta_2(0) \theta_4(0)} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \end{aligned} \tag{3.16}$$

One can give a more symmetric form to these results. Using the standard identities⁽²⁵⁾

$$\begin{aligned} \theta_1(z) &= \frac{2\theta_1(z/2)\theta_2(z/2)\theta_3(z/2)\theta_4(z/2)}{\theta_2(0)\theta_3(0)\theta_4(0)} \\ \theta_1(z, 2\tau) &= \frac{\theta_1(z/2)\theta_2(z/2)}{\theta_4(0, 2\tau)} \\ \theta_4(z, 2\tau) &= \frac{\theta_3(z/2)\theta_4(z/2)}{\theta_4(0, 2\tau)} \tag{3.17} \\ \theta_2\left(z, \frac{\tau}{2}\right) &= \left[\frac{2}{\theta_2(0)\theta_3(0)}\right]^{1/2} \theta_2(z)\theta_3(z) \\ \theta_1\left(z, \frac{\tau}{2}\right) &= \left[\frac{2}{\theta_2(0)\theta_3(0)}\right]^{1/2} \theta_1(z)\theta_4(z) \end{aligned}$$

and characterizing a twisted sector by the index $\nu = 2, \dots, 4$ through (2.19), one finds

$$\begin{aligned} \langle O_{EM}(1) O_{-E, -M}(2) \rangle_{(\alpha\beta) \neq (00)} \leftrightarrow \nu \\ = \left[\frac{\theta'_1(0)}{\theta_1(z_{12})} \prod_{\substack{\nu'=2 \\ \nu' \neq \nu}}^4 \frac{\theta_{\nu'}^2(z_{12}/2)}{\theta_{\nu'}^2(0)} \right]^{2h_{EM}} [\text{cc}]^{2h_{EM}} \end{aligned} \tag{3.18}$$

One can check that formulas (3.14)–(3.16) satisfy appropriate modular transformation properties

$$\begin{aligned} \langle O_{EM} O_{-E, -M} \rangle_{(0, 1/2)}(z_{12}, \tau) \\ = z_{12}^{-2h_{EM}} \bar{z}_{12}^{-2h_{EM}} \langle O_{EM} O_{-E, -M} \rangle_{(1/2, 0)}(z_{12}/\tau, -1/\tau) \end{aligned} \tag{3.19a}$$

$$\langle O_{EM} O_{-E, -M} \rangle_{(1/2, 1/2)}(z_{12}, \tau) = \langle O_{EM} O_{-E, -M} \rangle_{(1/2, 0)}(z_{12}, \tau + 1) \tag{3.19b}$$

3. The total correlation function is finally

$$\begin{aligned} \langle O_{EM}(1) O_{-E, -M}(2) \rangle \\ = \frac{1}{2} Z_C \langle O_{EM} O_{-E, -M} \rangle_C + \frac{\sum_{(\alpha\beta) \neq (00)} Z_{(\alpha\beta)} \langle O_{EM} O_{-E, -M} \rangle_{(\alpha\beta)}}{Z^{\text{AT}}} \end{aligned} \tag{3.20}$$

It is in general not periodic. Note that in (3.20), once the $z_{12}^{-2h_{EM}} [\text{cc}]^{-2h_{EM}}$ are factorized, all the remaining terms are integer powers of z_{12}, \bar{z}_{12} . This is due to the fact that in the short-distance expansion only derivatives of the field φ appear that have integer dimensions.

We now discuss a few applications. As explained in Section 2, the case $E = 1$ (resp. $E = -1$), $M = \frac{1}{2}$ corresponds to the fermions ψ ($\bar{\psi}$) in the Luttinger model (2.10). They become analytic (resp. antianalytic) at $g = 2$, where the four fermion coupling Γ vanishes, and the AT model decouples

into two independent Ising (Z_2) models. In this case $h_{1,1/2} = 1/2$, $\bar{h}_{1,1/2} = 0$; thus,

$$\begin{aligned} \frac{Z_C}{2} \langle \psi(z_1) \psi^+(z_2) \rangle_C &= \frac{1}{(2\tau_1)^{1/2} |\eta|^2} \frac{\theta'_1(0)}{\theta_1(z_{12})} \exp\left(-\frac{\pi}{2\tau_1} z_{12}^2\right) \\ &\times \sum_{mm' \in \mathbb{Z}} \exp\left[2\pi \frac{m' - m\bar{c}}{\tau_1} z_{12} - 2\pi \frac{|m' - m\tau|^2}{\tau_1}\right] \end{aligned} \quad (3.21)$$

which is easily proven to be

$$\frac{Z_C}{2} \langle \psi(z_1) \psi^+(z_2) \rangle_C = \frac{1}{4 |\eta|^2} \frac{\theta'_1(0)}{\theta_1(z_{12})} \sum_{v=2}^4 \overline{\theta_v(0)} \theta_v(z_{12}) \quad (3.22)$$

using the definition of theta functions and the identity (4.19) of Ref. 14. In the sector $(\alpha\beta) = (\frac{1}{2}, 0)$ one finds

$$\langle \psi(z_1) \psi^+(z_2) \rangle_{(1/2,0)} = \frac{\theta_2(z_{12}/2, \tau/2) \theta'_1(0, \tau/2)}{2\theta_1(z_{12}/2, \tau/2) \theta_2(0, \tau/2)} \quad (3.23)$$

Using (3.17) and Ref. 25,

$$\begin{aligned} \theta_2(z) &= \frac{\theta_2^2(z/2, \tau/2) - \theta_1^2(z/2, \tau/2)}{2\theta_3(0)} \\ \theta_3(z) &= \frac{\theta_2^2(z/2, \tau/2) + \theta_1^2(z/2, \tau/2)}{2\theta_2(0)} \end{aligned} \quad (3.24)$$

this reads

$$\langle \psi(z_1) \psi^+(z_2) \rangle_{(1/2,0)} = \frac{\theta'_1(0)}{\theta_1(z_{12})} \left[\frac{\theta_2(z_{12})}{2\theta_2(0)} + \frac{\theta_3(z_{12})}{2\theta_3(0)} \right] \quad (3.25)$$

Modular transformation then gives similar results in the other sectors. Combining (3.22) and (3.25), one gets

$$\langle \psi(z_1) \psi^+(z_2) \rangle^{(2)} = \frac{\theta'_1(0)}{\theta_1(z_{12})} \frac{\sum_{v=2}^4 [|\theta_v(0)|/\theta_v(0)] \theta_v(z_{12})}{\sum_{v=2}^4 |\theta_v(0)|} \quad (3.26)$$

A comparison with (2.22) allows one to identify the fermion correlation function for the Ising model in a fermionic sector v ,

$$\langle \psi(z_1) \psi^+(z_2) \rangle_v^{(1)} = \frac{\theta'_1(0)}{\theta_1(z_{12})} \frac{\theta_v(z_{12})}{\theta_v(0)}, \quad v = 2, \dots, 4 \quad (3.27)$$

in agreement with the result of direct calculations.⁽¹⁴⁾

In a similar way the point $g = 3$ corresponds to the Z_4 model of Ref. 16, for which quarter-integer-spin parafermions χ (ψ_1 in Ref. 16) are described by $O_{3/2,1/2}$ with $h_{3/2,1/2} = 3/4$, $\bar{h}_{3/2,1/2} = 0$. We thus get

$$\begin{aligned} & \langle \chi(z_1) \chi^+(z_2) \rangle^{(Z_4)} \\ &= \frac{1}{Z^{\Delta T}} \left[\frac{\theta'_1(0)}{\theta_1(z_{12})} \right]^{3/2} \left\{ \frac{1}{2|\eta|^2} \left(\frac{3}{\tau_I} \right)^{1/2} \exp \left(-\frac{3\pi}{4\tau_I} z_{12}^2 \right) \right. \\ & \quad \times \sum_{mm' \in Z} \exp \left(3\pi \frac{m' - m\bar{\tau}}{\tau_I} z_{12} - 3\pi \frac{|m' - m\tau|^2}{\tau_I} \right) \\ & \quad \left. + \frac{|\eta|}{|\theta_2(0)|} \left[\frac{\theta_3(z_{12}/2) \theta_4(z_{12}/2)}{\theta_3(0) \theta_4(0)} \right]^3 + \text{perm.} \right\} \end{aligned} \tag{3.28}$$

Whether (3.28) can be decomposed as a sum over “parafermionic sectors” similar to (3.26), (3.27) is unclear.

4. Most of the correlation functions must be periodic, however, which is not the case for (3.20), even for E, M integers, because of the twisted sector contribution. To represent the energy operator (discussed in Section 2), one must thus consider $\cos 2\varphi$ instead of $e^{2i\varphi}$. The distinction between these two possibilities has no meaning on the plane, but becomes relevant on the torus. We write, for instance,

$$\begin{aligned} & 2 \langle \cos E\varphi(1) \cos E\varphi(2) \rangle_{(1/2,0)} \\ &= \left| \frac{\theta'_1(0, \tau/2)}{\theta_2(0, \tau/2)} \right|^{E^2/g} \left\{ \left| \frac{\theta_2(z_{12}, \tau/2)}{\theta_1(z_{12}, \tau/2)} \right|^{E^2/g} + \left| \frac{\theta_1(z_{12}, \tau/2)}{\theta_2(z_{12}, \tau/2)} \right|^{E^2/g} \right\} \end{aligned} \tag{3.29}$$

This expression corresponds to giving a meaning to all the terms in the contraction using Wick’s theorem, even those for which the electric neutrality is broken, which is possible since there is no zero mode in a twisted sector, as we discuss later. Using (3.9) and (3.18), one has

$$\begin{aligned} & 2 \langle \cos E\varphi(1) \cos E\varphi(2) \rangle \\ &= \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{E^2/g} \left\{ \frac{1}{2|\eta|^2} \left(\frac{g}{\tau_I} \right)^{1/2} \exp \left(\frac{\pi E^2 \text{Im}^2 z_{12}}{g \tau_I} \right) \right. \\ & \quad \times \sum_{mm' \in Z} \exp \left[2i\pi E \left(m \text{Re} z_{12} - \frac{m\tau_R - m'}{\tau_I} \text{Im} z_{12} \right) - \pi g \frac{(m\tau - m')^2}{\tau_I} \right] \\ & \quad + \frac{|\eta|}{|\theta_2(0)| |\theta_3(0) \theta_4(0)|^{2E^2/g}} \\ & \quad \left. \times \left[\left| \theta_1 \left(\frac{z_{12}}{2} \right) \theta_2 \left(\frac{z_{12}}{2} \right) \right|^{2E^2/g} + \left| \theta_3 \left(\frac{z_{12}}{2} \right) \theta_4 \left(\frac{z_{12}}{2} \right) \right|^{2E^2/g} \right] + \text{perm.} \right\} \end{aligned} \tag{3.30}$$

and $E=2$ gives the energy correlation function. In the same way $E=1$ corresponds to the polarization operator, and also, at the Z_4 point, to the operator σ_2 in the notation of Ref. 16.

5. There is no zero mode in a twisted sector, and it is thus possible to give a (renormalized) meaning to $\langle \varphi^2 \rangle_{(\alpha\beta) \neq (00)}$, which diverges logarithmically. Introducing a mass term $m^2 \varphi^2$ in (2.6), following a zeta regularization and then letting m go to zero, we find⁽¹³⁾

$$\langle e^{iE\varphi} \rangle_{(0,1/2)} = \exp(-\frac{1}{2}E^2 \langle \varphi^2 \rangle_{(0,1/2)}) = [\frac{1}{2}\pi |\theta_2(0)|^2]^{E^2/2g} \quad (3.31a)$$

and similar results obtained by modular transformations

$$\langle e^{iE\varphi} \rangle_{(1/2,0)} = [\frac{1}{2}\pi |\theta_4(0)|^2]^{E^2/2g} \quad (3.31b)$$

$$\langle e^{iE\varphi} \rangle_{(1/2,1/2)} = [\frac{1}{2}\pi |\theta_3(0)|^2]^{E^2/2g} \quad (3.31c)$$

The modular dependence in (3.31) is expected to be universal, unlike the numerical factors, which depend on the regularization. From (3.31) we deduce, for instance, the finite-size effects for the mean value of the energy operator

$$\langle \varepsilon \rangle = \left(\frac{\pi}{2}\right)^{2/g} \frac{|\eta|}{Z_{\text{AT}}} \sum_{v=2}^4 |\theta_v(0)|^{4/g-1} \quad (3.32)$$

and at the decoupling point $g=2$

$$\langle \varepsilon \rangle = 2\pi \frac{|\eta|^3}{\sum_{v=2}^4 |\theta_v(0)|} \quad (3.33)$$

a result which agrees with asymptotic analysis of the exact (lattice) solution of the Ising model⁽²⁶⁾ as well as with direct calculations involving fermions⁽¹³⁾ (the energy being represented by $\psi\tilde{\psi}$ in the latter case) and justifies the choice of numerical constants in (3.31). Using the same identities as before, one can check⁽¹⁴⁾ that for $E=2, g=2$

$$2\langle \cos 2\varphi(1) \cos 2\varphi(2) \rangle = \langle \psi\tilde{\psi}(1) \psi^+ \tilde{\psi}^+(2) \rangle^{(1)} + (\langle \varepsilon \rangle^{(1)})^2 \quad (3.34)$$

where the fermionic propagator is given by (3.25). The additional term comes from the fact that $\cos 2\varphi$ represents the sum of the energies of the two independent Ising models and thus

$$2\langle \cos 2\varphi \cos 2\varphi \rangle = 1/2\langle (\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_2) \rangle = \langle \varepsilon\varepsilon \rangle + \langle \varepsilon \rangle^2$$

We can also give a meaning to isolated vortices in a twisted sector, since antiperiodic boundary conditions now allow a frustration line (starting at an isolated vortex) to close onto itself (Fig. 6). Since the duality

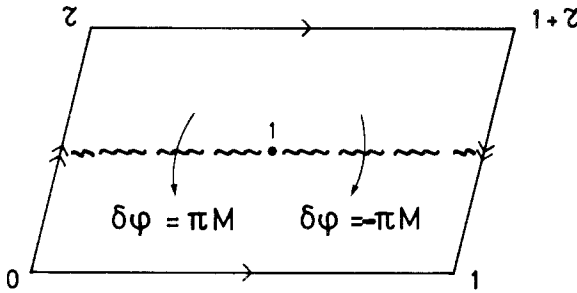


Fig. 6. In a twisted sector one can give a meaning to an isolated vortex, since antiperiodicity (along ω' here) allows a frustration line to close onto itself.

for the Gaussian model leaves a twisted sector unchanged, their mean value is readily calculated as

$$\langle O_M \rangle_{(\alpha\beta)}(g) = \langle e^{iM\varphi} \rangle_{(\alpha\beta)}(1/g) \tag{3.35}$$

It is nevertheless instructive to rederive this result directly. We consider, for instance, $(\alpha\beta) = (\frac{1}{2}, 0)$. To take into account the presence of an isolated vortex at, say, $z = 0$, we introduce a real classical field such that

$$\partial_z \varphi_{cl} = -i \frac{M \theta'_1(0) \theta_4(z)}{2 \theta_4(0) \theta_1(z)} \tag{3.36}$$

Due to⁽²⁷⁾

$$\int dz \frac{\theta_4(z)}{\theta_1(z)} = \frac{\theta_4(0)}{\theta'_1(0)} \log \left[\frac{\theta_1(z)}{\theta_2(z) \theta_3(0) + \theta_2(0) \theta_3(z)} \right] \tag{3.37}$$

one has

$$\varphi_{cl} = \frac{-iM}{2} \log \left[\frac{\theta_1(z)}{\theta_2(z) \theta_3(0) + \theta_2(0) \theta_3(z)} \right] + cc \tag{3.38}$$

φ_{cl} satisfies the boundary conditions $\varphi_{cl}(z+1) = -\varphi_{cl}(z)$, $\varphi_{cl}(z+\tau) = \varphi_{cl}(z)$, and presents the desired discontinuity around $z = 0$, since

$$\int_{\mathcal{C} \text{ surrounding } 0} \partial_z \varphi_{cl} dz + \partial_{\bar{z}} \varphi_{cl} d\bar{z} = 2\pi M \tag{3.39}$$

by simple application of the residue theorem or (3.37). We thus find the weight of an isolated vortex to be formally

$$\langle O_M \rangle_{(1/2,0)} = \exp \left[-\frac{g}{4\pi} M^2 \left| \frac{\theta'_1(0)}{\theta_4(0)} \right|^2 \int_r \left| \frac{\theta_4(z)}{\theta_1(z)} \right|^2 d^2x \right] \tag{3.40}$$

As expected, the integral in the exponential diverges logarithmically. A regularized value of such integrals was obtained in Ref. 13 by a rather indirect procedure. Adding a mass term to the Ising action, we have calculated the partition function in each fermionic sector with a zeta regularization, obtaining

$$Z_\nu(m) = Z_\nu(m=0) \{ 1 - A\pi m^2 \log[\frac{1}{2}\pi|\theta_\nu(0)|^2] + O(m^4) \} \quad (3.41)$$

(A is the area of the torus). The m^2 term is in agreement with the lattice calculations for the specific heat in Ref. 26; on the other hand, it is given by the integral over the torus of the modulus square of the fermionic propagator, i.e., precisely due to (3.27), the integral in (3.40). Thus, a finite renormalized value is

$$\int_T \left| \frac{\theta_4(z)}{\theta_1(z)} \right|^2 d^2x = -2\pi \left| \frac{\theta_4(0)}{\theta_1'(0)} \right|^2 \log \left(\frac{\pi}{2} |\theta_4(0)|^2 \right) \quad (3.42)$$

and

$$\langle O_M \rangle_{(1/2,0)} = [\frac{1}{2}\pi |\theta_4(0)|^2]^{gM^2/2} \quad (3.43)$$

in agreement with (3.35). Finally, for a more general O_{EM} operator we find, suppressing as usual the logarithmic divergence of φ_{cl} as $z \rightarrow 0$,

$$\langle O_{EM} \rangle_{(1/2,0)} = [\frac{1}{2}\pi\theta_4^2(0)]^{h_{EM}} [\text{cc}]^{h_{EM}} \quad (3.44)$$

and similar results in the other sectors obtained by modular transformations.

6. So far we have discussed the two-point functions only, but the same methods can be applied to the calculation of multipoint correlators without any further difficulty. For a collection of operators $O_{E_1M_1}(1) \cdots O_{E_nM_n}(n)$ such that $\sum E_i = \sum M_i = 0$, the classical field analogous to (3.1) reads

$$\varphi_{cl} = \sum_{i=1}^n M_i [\text{Im} \log \theta_1(z - z_i) - (2\pi/\tau_1) \text{Im} z \text{Re} z_i] \quad (3.45)$$

and with the use of Wick's theorem one finds

$$\begin{aligned} &\langle O_{E_1M_1}(1) \cdots O_{E_nM_n}(n) \rangle_{pp} \\ &= \prod_{j < k} \left| \frac{\theta_1(z_{jk})}{\theta_1'(0)} \right|^{E_j E_k / g + g M_j M_k} \\ &\quad \times \exp \left[\frac{\pi}{g\tau_1} \left(\sum_j E_j \text{Im} z_j - ig M_j \text{Re} z_j \right)^2 + 2i \sum_{j < k} E_j M_k \text{Im} \log \frac{\theta_1(z_{jk})}{\theta_1'(0)} \right] \\ &= \prod_{j < k} \left[\frac{\theta_1(z_{jk})}{\theta_1'(0)} \right]^{2\delta_j \delta_k} [\text{cc}]^{2\delta_j \delta_k} \exp \left[-\frac{\pi}{\tau_1} \left(\sum_j \delta_j z_j + \delta_j \bar{z}_j \right)^2 \right] \end{aligned} \quad (3.46)$$

where $\delta_j = \delta_{E_j M_j}$, while

$$\begin{aligned} & \langle O_{E_1 M_1}(1) \cdots O_{E_n M_n}(n) \rangle_{mm'} \\ &= \langle O_{E_1 M_1}(1) \cdots O_{E_n M_n}(n) \rangle_{pp} \exp \left[2\pi g^{1/2} \left(\frac{m' - m\bar{\tau}}{\tau_1} \right. \right. \\ & \quad \left. \left. \times \sum_j \delta_j z_j + \frac{m' - m\tau}{\tau_1} \sum_j \delta_j \bar{z}_j \right) \right] \end{aligned} \tag{3.47}$$

In the twisted sector $(\alpha\beta) = (\frac{1}{2}0)$, we have also

$$\varphi_{cl} = \sum_i M_i \operatorname{Im} \log \left[\frac{\theta_1((z - z_i)/2, \tau/2)}{\theta_2((z - z_i)/2, \tau/2)} \right] \tag{3.48}$$

Thus

$$\begin{aligned} & \langle O_{E_1 M_1}(1) \cdots O_{E_n M_n}(n) \rangle_{(1/2, 0)} \\ &= \prod_{j < k} \left[\frac{2\theta_1(z_{jk}/2, \tau/2) \theta_2(0, \tau/2)}{\theta_2(z_{jk}/2, \tau/2) \theta'_1(0, \tau/2)} \right]^{2\delta_j \delta_k} [\text{cc}]^{2\delta_j \delta_k} \end{aligned} \tag{3.49}$$

and similar results obtained by modular transformations. Note that the global normalization of these n -point functions, which is usually defined recursively⁽¹⁴⁾ by short-distance expansions, was here very easy to write due to⁽¹⁹⁾

$$O_{E_1 M_1}(1) O_{E_2 M_2}(2) \sim z_{12}^{2\delta_1 \delta_2} \bar{z}_{12}^{2\delta_1 \delta_2} O_{E_1 + E_2, M_1 + M_2}(1) \tag{3.50}$$

since

$$h_{E_1 + E_2, M_1 + M_2} - h_{E_1 M_1} - h_{E_2 M_2} = 2\delta_1 \delta_2$$

4. TWIST OPERATOR CORRELATION FUNCTIONS

As explained in the introduction, there are interesting physical quantities, such as the spin, which are not described by operators of the preceding O_{EM} type, but instead by twist operators, which create a branch point singularity with change of sign of the field φ . Their correlation functions are technically more difficult to obtain, and we shall generalize in the following the methods proposed in Ref. 18 for the corresponding problem in the plane. A similar approach has also been used for the calculation of Ising spin correlations in Refs. 28 and 14.

1. We first consider the two-point function in the doubly periodic sector for the basic twist field, which obeys⁽¹⁸⁾

$$\partial_z \varphi(z) \sigma(z_1, \bar{z}_1) \sim (z - z_1)^{-1/2} \tau(z_1, \bar{z}_1) \tag{4.1}$$

(here τ is another “excited” twist field) and, as proposed in Ref. 18, we introduce the auxiliary function

$$G(z, w, z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{\langle -g \partial_z \varphi(z) \partial_w \varphi(w) \sigma(1) \sigma(2) \rangle}{\langle \sigma(1) \sigma(2) \rangle} \tag{4.2}$$

which can be completely determined using local monodromy and consistency requirements. Indeed, G must be doubly periodic and analytic in the z, w variables and present the short-distance behaviors

$$G \sim \frac{1}{(z - z_i)^{1/2}} \left(\text{resp} \sim \frac{1}{(w - z_i)^{1/2}} \right) \quad \text{as } z \text{ (resp } w) \rightarrow z_i \tag{4.3}$$

$$G = \frac{1}{2(z - w)^2} + \text{regular terms as } z \rightarrow w$$

A natural candidate is then

$$G^{(0)} = \left[\frac{\theta'_1(0)}{2\theta_1(z - w)} \right]^2 \left\{ \left[\frac{\theta_1(z - z_1) \theta_1(w - z_2)}{\theta_1(z - z_2) \theta_1(w - z_1)} \right]^{1/2} \times \left[\frac{\theta_1(z - w + z_{12}/4)}{\theta_1(z_{12}/4)} \right]^2 + z_1 \leftrightarrow z_2 \right\} \tag{4.4}$$

A given determination of the square roots in the θ plane is chosen, which translates into a cut relating z_1 to z_2 in the z plane, and it can be checked that $G^{(0)}$ is periodic. However, (4.4) is not the only possible choice. If we introduce the auxiliary function

$$\Omega_1(z, z_1, z_2) = \frac{\theta_1(z - (z_1 + z_2)/2)}{[\theta_1(z - z_1) \theta_1(z - z_2)]^{1/2}} \tag{4.5}$$

it is clear that

$$G = G^{(0)} + \lambda(1, 2) \Omega_1(z) \Omega_1(w) \tag{4.6}$$

where λ is an arbitrary function, also satisfies the local conditions (4.3). Conversely, if two possible functions $G^{(1)}$ and $G^{(2)}$ are given, the difference $G^{(1)} - G^{(2)}$ is regular as $z \rightarrow w$. Forming the ratio $(G^{(1)} - G^{(2)})/\Omega_1(z) \Omega_1(w)$ then removes the branch point singularities, and leaves an analytic, doubly periodic function in z and w with at most a *single* pole in $z(w) = (z_1 + z_2)/2$; it is thus a constant⁽²⁷⁾ and (4.6) is in fact the most general solution of the problem. In order to fix $\lambda(1, 2)$, we can now use the fact that the field φ must be uniquely defined on the torus (as long as one remains on a given

sheet); this was called a global monodromy requirement in Ref. 18. We need

$$H(\bar{z}, w, z_1, \bar{z}_1, z_1, \bar{z}_2) = \frac{\langle -g \partial_z \varphi(\bar{z}) \partial_w \varphi(w) \sigma(1) \sigma(2) \rangle}{\langle \sigma(1) \sigma(2) \rangle} \tag{4.7}$$

which obeys conditions similar to (4.3), but without singularity as $z \rightarrow w$; we thus write

$$H = \mu(1, 2) \bar{\Omega}_1(\bar{z}) \Omega_1(w) \tag{4.8}$$

Then we must impose for any contour \mathcal{C} that does not cross the cut

$$\int_{\mathcal{C}} \partial_z \varphi(z) dz + \partial_{\bar{z}} \varphi(\bar{z}) d\bar{z} = 0$$

which translates into

$$\int_{\mathcal{C}} G dz + H d\bar{z} = 0 \tag{4.9}$$

Relation (4.9) is trivially true for any contractible \mathcal{C} that does not surround z_1, z_2 , by application of residue theorem. If \mathcal{C} surrounds the cut, it can be deformed to the parallelogram boundary of the torus, and the integral still vanishes, since G, H are periodic. We are thus left with noncontractible contours, which give the same integrals in a given homotopy class, i.e., $\mathcal{C} \equiv \omega$ or ω' . This gives two conditions, fixing in principle λ and μ .

Suppose $\lambda(1, 2)$ is known. Then we can take the limit $z \rightarrow w$ in (4.6), which, due to⁽¹⁴⁾

$$T(z) = -g : (\partial_z \varphi)^2 := - \lim_{z \rightarrow w} \left[g \partial_z \varphi(z) \partial_w \varphi(w) + \frac{1}{2(z-w)^2} \right] \tag{4.10}$$

gives

$$\begin{aligned} \frac{\langle T(z) \sigma(1) \sigma(2) \rangle}{\langle \sigma(1) \sigma(2) \rangle} &= \frac{1}{16} \left[\frac{\theta'_1}{\theta_1}(z-z_1) - \frac{\theta'_1}{\theta_1}(z-z_2) \right]^2 \\ &+ \frac{1}{2} \frac{\theta'_1}{\theta_1} \left(\frac{z_{12}}{4} \right) \left[\frac{\theta'_1}{\theta_1}(z-z_1) - \frac{\theta'_1}{\theta_1}(z-z_2) \right] \\ &+ \lambda(1, 2) \frac{\theta_1^2(z - (z_1 + z_2)/2)}{\theta_1(z-z_1)\theta_1(z-z_2)} + \text{terms regular as } z \rightarrow z_1 \end{aligned} \tag{4.11}$$

Now, since $T(z)$ is the generator of conformal transformations, it satisfies, as $z \rightarrow z_1$,⁽⁶⁾

$$T(z) \sigma(z_1, \bar{z}_1) = \frac{h_\sigma}{(z - z_1)^2} + \frac{\partial_{z_1} \sigma(z_1, \bar{z}_1)}{z - z_1} + \text{terms regular as } z \rightarrow z_1 \quad (4.12)$$

Thus, letting $z \rightarrow z_1$ in (4.11), we obtain $h_\sigma = 1/16$, as expected,⁽¹⁸⁾ and

$$\begin{aligned} \partial_{z_1} \log \langle \sigma(1) \sigma(2) \rangle_{pp} &= -\frac{1}{8} \frac{\theta_1'}{\theta_1}(z_{12}) + \frac{1}{2} \frac{\theta_1'}{\theta_1} \left(\frac{z_{12}}{4} \right) \\ &+ \lambda(1, 2) \frac{\theta_1^2(z_{12}/2)}{\theta_1'(0) \theta_1(z_{12})} \end{aligned} \quad (4.13)$$

To determine λ , we first multiply the integrals in (4.9) by $\theta_1^2(z_{12}/2)/[\theta_1'(0) \theta_1(z_{12}) \Omega_1(w)]$ and choose for w the value $w = z_1$. Now consider the function

$$\theta_1^2(z_{12}/2) G^{(0)}(z, w = z_1, 1, 2)/[\theta_1'(0) \theta_1(z_{12}) \Omega_1(w = z_1)]$$

It is analytic and doubly periodic as a function of z with $(z - z_1)^{-1/2}$ and $(z - z_1)^{-3/2}$ singularities as $z \rightarrow z_1$. It can thus be written as a linear combination of $\Omega_1(z, z_1, z_2)$ and $\partial_{z_1} \Omega_1(z, z_1, z_2)$, the coefficients of which are determined by studying the $z \rightarrow z_1$ behavior,

$$\begin{aligned} &\frac{\theta_1^2(z_{12}/2) G^{(0)}(z, w = z_1, 1, 2)}{\theta_1'(0) \theta_1(z_{12}) \Omega_1(w = z_1)} \\ &= \frac{1}{2} \frac{\theta_1'}{\theta_1} \left(\frac{z_{12}}{4} \right) \Omega_1(z, z_1, z_2) + \frac{1}{2} \partial_{z_1} \Omega_1(z, z_1, z_2) \end{aligned} \quad (4.14)$$

Condition (4.9) then reads

$$\begin{aligned} \int_{\omega(\omega')} &\left\{ \left[\frac{\lambda(1, 2) \theta_1^2(z_{12}/2)}{\theta_1'(0) \theta_1(z_{12})} + \frac{1}{2} \frac{\theta_1'}{\theta_1} \left(\frac{z_{12}}{4} \right) \right] \Omega_1(z, z_1, z_2) \right. \\ &\left. + \frac{1}{2} \partial_{z_1} \Omega_1(z, z_1, z_2) \right\} dz + \frac{\mu(1, 2) \theta_1^2(z_{12}/2)}{\theta_1'(0) \theta_1(z_{12})} \bar{\Omega}_1(\bar{z}, \bar{z}_1, \bar{z}_2) d\bar{z} = 0 \end{aligned} \quad (4.15)$$

which is easily solved, giving

$$\begin{aligned} &\partial_{z_1} \log \langle \sigma(1) \sigma(2) \rangle_{pp} \\ &= \partial_{z_1} \log \theta_1^{-1/8}(z_{12}) \\ &+ \frac{1}{2} \frac{\int_{\omega'} \partial_{z_1} \Omega_1 dz \int_{\omega} \bar{\Omega}_1 d\bar{z} - \int_{\omega} \partial_{z_1} \Omega_1 dz \int_{\omega'} \bar{\Omega}_1 d\bar{z}}{\int_{\omega} \Omega_1 dz \int_{\omega'} \bar{\Omega}_1 d\bar{z} - \int_{\omega'} \Omega_1 dz \int_{\omega} \bar{\Omega}_1 d\bar{z}} \end{aligned} \quad (4.16)$$

Introducing

$$I_1(I'_1) = \int_{\omega(\omega')} \Omega_1(z, z_1, z_2) dz \tag{4.17}$$

we find then

$$\langle \sigma(1) \sigma(2) \rangle_{pp} \propto \frac{1}{[\theta_1(z_{12})]^{1/8}} \frac{1}{[\text{Im}(\bar{I}_1 I'_1)]^{1/2}} \tag{4.18}$$

The unknown term depending on \bar{z}_{12} in (4.18) clearly must be $[\theta_1(z_{12})]^{-1/8}$. The normalization can be fixed by looking at $z_1 \rightarrow z_2$ behavior. Then $\Omega_1 \rightarrow 1, I_1 \rightarrow 1, I'_1 \rightarrow \tau$, and thus

$$\langle \sigma(1) \sigma(2) \rangle_{pp} = \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \frac{\tau_1^{1/2}}{[\text{Im}(\bar{I}_1 I'_1)]^{1/2}} \tag{4.19}$$

It is not the modulus square of an analytic function, and it is not periodic.

2. Now we consider a soliton sector with (m, m') characteristics, and we introduce a real classical field such that

$$\partial_z \varphi_{cl} = A(1, 2) \Omega_1(z, z_1, z_2) \tag{4.20}$$

which satisfies the correct monodomy conditions around z_1, z_2 , and fix A by the condition

$$\int_{\omega(\omega')} \partial_z \varphi_{cl} dz + \partial_{\bar{z}} \varphi_{cl} d\bar{z} = 2\pi m(2\pi m') \tag{4.21}$$

which is readily solved,

$$A(1, 2) = i\pi \frac{m\bar{I}'_1 - m'I_1}{\text{Im}(\bar{I}_1 I'_1)} \tag{4.22}$$

Writing $\varphi \rightarrow \varphi + \varphi_{cl}$, where φ is now periodic, the two contributions factorize as usual. The classical action reads, from (2.14).

$$\mathcal{A}_{cl} = \pi g \frac{(mI'_1 - m'I_1)^2}{[\text{Im}(\bar{I}_1 I'_1)]^2} \int_T |\Omega_1|^2 d^2x \tag{4.23}$$

Actually, this expression can be simplified. We can consider $\int_T |\Omega_1|^2 d^2x$ in the simply connected region \tilde{T} whose boundary is the parallelogram representing the torus, related by a thin neck to a contour surrounding the cut (Fig. 7). In \tilde{T} , there exists a function such that $df = \Omega_1 dz$. Using

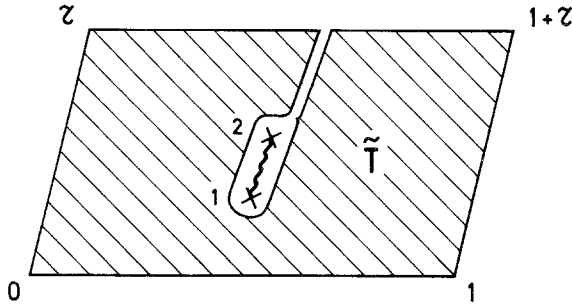


Fig. 7. To evaluate the integral in (4.23), we consider the simply connected region \tilde{T} whose boundary is the usual parallelogram representing the torus related by a thin neck to a contour surrounding the cut.

Green's theorem and integrating in a symmetric way along ω , $\omega + \tau$ (resp. ω' , $\omega' + 1$), we get

$$\begin{aligned} \int_T |\Omega_1|^2 d^2x &= \frac{1}{2i} \int_{\tilde{T}} \Omega_1 dz \wedge \bar{\Omega}_1 d\bar{z} \\ &= \frac{1}{2i} \left[\int_{\omega} \bar{\Omega}_1 \int_{\omega'} \Omega_1 - \int_{\omega} \Omega_1 \int_{\omega'} \bar{\Omega}_1 \right] \\ &= \text{Im}(\bar{I}_1 I_1) \end{aligned} \tag{4.24}$$

the contribution of the contour surrounding the cut disappearing because, as explained above, the corresponding integral of Ω_1 vanishes. As $z_1 \rightarrow z_2$, \mathcal{A}_{cl} becomes naturally $\pi g |m\tau - m'|^2/\tau_1$, so the correctly normalized result takes the form

$$\begin{aligned} Z_{mm'} \langle \sigma(1) \sigma(2) \rangle_{mm'} &= \frac{\sqrt{g}}{|\eta|^2} \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \frac{1}{[\text{Im}(\bar{I}_1 I_1)]^{1/2}} \\ &\times \exp \left[-\pi g \frac{|mI'_1 - m'I_1|^2}{\text{Im}(\bar{I}_1 I_1)} \right] \end{aligned} \tag{4.25}$$

3. The results in the other sectors can then be obtained by translating z_{12} by 1 , τ , or $1 + \tau$, which wraps a twist line around the torus. The relevant contours of integration are still homotopic of ω , ω' , passing possibly twice through the cut (Fig. 8). Along these contours there is no change of sign for the field φ , which can now present the usual discontinuities of $2\pi m$, $2\pi m'$. The distinction we made in the first section between Coulombic and twisted sectors becomes in fact irrelevant. All these sectors are related by

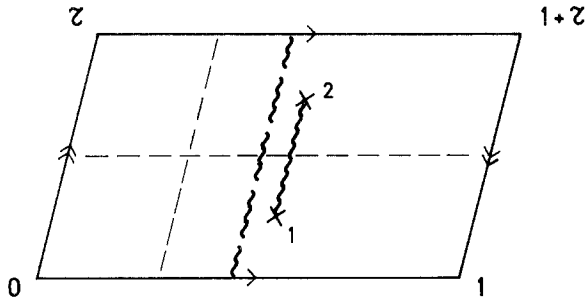


Fig. 8. The $\langle \sigma\sigma \rangle$ correlation function in the $(\alpha\beta) = (\frac{1}{2}, 0)$ sector is obtained shifting z_{12} by τ . Then one of the two contours used in the global monodromy requirement passes twice through the cut.

the condition of periodicity of $\langle \sigma\sigma \rangle$, which imposes to sum over the different translations of z_{12} with the same weight. Introducing

$$\Omega_\nu(z, z_1, z_2) = \frac{\theta_\nu(z - (z_1 + z_2)/2)}{[\theta_1(z - z_1) \theta_1(z - z_2)]^{1/2}} \tag{4.26}$$

and

$$I_\nu(I'_\nu) = \int_{\omega(\omega')} \Omega_\nu dz \tag{4.27}$$

we get

$$\begin{aligned} \langle \sigma(1) \sigma(2) \rangle &= \frac{1}{Z^{\text{AT}}} \frac{\sqrt{g}}{2|\eta|^2} \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \\ &\times \sum_{\nu=1}^4 \frac{1}{[\text{Im}(\bar{I}_\nu I'_\nu)]^{1/2}} \sum_{mm' \in \mathbb{Z}} \exp \left[-\pi g \frac{|mI'_\nu - m'I_\nu|^2}{\text{Im}(\bar{I}_\nu I'_\nu)} \right] \end{aligned} \tag{4.28}$$

Note that $\langle \sigma\sigma \rangle$ depends on g in all sectors, which was not the case of the partition function, whose twisted contribution (2.19) was independent of g .

It is interesting to consider the short-distance behavior of this expression. In the $(\alpha\beta) = (\frac{1}{2}, 0)$ sector, for instance, one has, using (.37),

$$I_4 = i\pi\theta_4(0)/\theta'_1(0) + O(z_{12}), \quad z_{12} \rightarrow 0 \tag{4.29}$$

while I_4 diverges logarithmically

$$\begin{aligned} I_4 &= \int_\omega \frac{\theta_4(z - (z_1 + z_2)/2)}{[\theta_4(z - z_1) \theta_4(z - z_2)]^{1/2}} \\ &= -\frac{2\theta_4(0)}{\theta'_1(0)} \log \frac{z_{12}}{4} - \frac{2\theta_4(0)}{\theta'_1(0)} \log \left[\frac{\pi}{2} \theta_4^2(0) \right] + O(z_{12}) \end{aligned} \tag{4.30}$$

in agreement with (3.42). Similar results are obtained in the other sectors, with the corresponding index ν for the theta function. Then, performing a Poisson transformation over m' in (4.28), one finds

$$\begin{aligned} \langle \sigma(1) \sigma(2) \rangle &= \frac{1}{Z^{\text{AT}}} \frac{1}{|z_{12}|^{1/4}} \left\{ \frac{Z_C}{2} + \sum_{\nu=2}^4 \frac{|\eta|}{|\theta_\nu(0)|} \right. \\ &\quad \times \left. \sum_{em \in 2\mathbb{Z}} \left[\frac{z_{12}}{4} \left(\frac{\pi}{2} \theta_\nu^2(0) \right) \right]^{h_{em}} [\text{cc}]^{h_{em}} + \dots \right\} \end{aligned} \quad (4.31)$$

where the dots denote terms with integer powers of z_{12}, \bar{z}_{12} . The finite contribution in (4.31) reproduces the partition function (2.20). In particular, the correct relative normalization between Coulombic and twisted sectors, which looked rather artificial in Section 2, is reproduced by the condition of periodicity for $\langle \sigma \sigma \rangle$. The $z_{12}^h \bar{z}_{12}^{\bar{h}}$ terms in (4.31) correspond to O_{EM} -type operators appearing in the short-distance expansion of $\sigma \sigma$. Identifying $[\frac{1}{2}\pi\theta_\nu^2(0)]^h [\text{cc}]^{\bar{h}}$ as $\langle O_{em} \rangle$, (3.44), we can read the structure constants

$$c_{\sigma\sigma, O_{em}} = 1/4^{x_{em}}, \quad em \in 2\mathbb{Z} \quad (4.32)$$

in agreement with the results of Refs. 17 and 18.

It is easy to verify using Poisson transformation that (4.28) presents the duality invariance

$$\langle \sigma(1) \sigma(2) \rangle(g) = \langle \sigma(1) \sigma(2) \rangle(1/g) \quad (4.33)$$

expected from (2.24), (2.25).

At the decoupling point $g=2$, $\langle \sigma \sigma \rangle$ must reproduce the Ising spin correlation function calculated in Ref. 14,

$$\langle S(1) S(2) \rangle^{(I)} = \left| \frac{\theta_1'(0)}{\theta_1(z_{12})} \right|^{1/4} \frac{\sum_{\nu=1}^4 |\theta_\nu(z_{12}/2)|}{\sum_{\nu=2}^4 |\theta_\nu(0)|} \quad (4.34)$$

Similarly, for $g=1$ we recover the XY model at the Kosterlitz–Thouless point, which is also described by a Coulombic partition function through the identity^(8,10)

$$Z^{\text{AT}}(g=1) = Z_c(g=4) = Z^{XY}$$

The spin correlation function is then simply obtained by considering $O_{1,0}$ in the model without twists,

$$\begin{aligned} \langle S(1) S(2) \rangle^{(XY)} &= \frac{1}{Z^{XY}} \frac{2}{\tau_1^{1/2} |\eta|^2} \left| \frac{\theta_1'(0)}{\theta_1(z_{12})} \right|^{1/4} \exp\left(\frac{\pi}{4\tau_1} \text{Im}^2 z_{12} \right) \\ &\quad \times \sum_{mm'} \exp \left\{ -4\pi \frac{|m' - m\tau|^2}{\tau_1} + 2\pi \frac{\text{Im}[(m' - m\bar{\tau}) z_{12}]}{\tau_1} \right\} \end{aligned} \quad (4.34\text{bis})$$

and this should equal (4.28) for $g = 1$. We discuss these identifications in Section 5.

Finally, let us note that excited twist correlators can be obtained from (4.1) by taking the limits $z \rightarrow z_1, w \rightarrow z_2$ in G , (4.6).

4. We can of course perform the same calculations for the $2n$ -point functions. We first consider the Coulombic sector and calculate the quantum part of the correlator for which the field presents n branch cuts $\varphi \rightarrow -\varphi$ and is otherwise single-valued. We introduce, as in (4.2),

$$G(z, w, 1, \dots, 2n) = \frac{\langle -g \partial_z \varphi(z) \partial_w \varphi(w) \sigma(1) \cdots \sigma(2n) \rangle}{\langle \sigma(1) \cdots \sigma(2n) \rangle} \tag{4.35}$$

with the monodromy constraints (4.3). The natural candidate generalizing (4.4) reads

$$\begin{aligned} G^{(0)} = & \left[2 \binom{2n}{n} \right]^{-1} \left[\frac{\theta_1'(0)}{\theta_1(z-w)} \right]^2 \left[\prod_{i=1}^{2n} \theta_1(z-z_i) \theta_1(w-z_i) \right]^{-1/2} \\ & \times \left\{ \sum_{i=1}^n \theta_1(z-z_i) \theta_1(w-z_{i+n}) \left[\frac{\theta_1(z-w + \sum_{i=1}^n \frac{1}{4}(z_i - z_{i+n}))}{\theta_1(\sum_{i=1}^n \frac{1}{4}(z_i - z_{i+n}))} \right]^2 \right. \\ & \left. + \text{permutations} \right\} \tag{4.36} \end{aligned}$$

On the other hand, the role of Ω_1 , (4.5), is now played by a set of n independent functions, a convenient choice of which is

$$\Omega_1^{(i)} = \frac{\theta_1[z-z_i + \frac{1}{2} \sum_{j=1}^n (z_j - z_{j+n})] \prod_{j=1: j \neq i}^n \theta_1(z-z_j)}{[\prod_{i=1}^{2n} \theta_1(z-z_i)]^{1/2}} \tag{4.37}$$

One proves easily that the most general slution for G is

$$G = G^{(0)} + \sum_{i,j=1}^n \lambda^{(ij)}(1, \dots, 2n) \Omega_1^{(i)}(z) \Omega_1^{(j)}(w) \tag{4.38}$$

For H defined by

$$\begin{aligned} H(\bar{z}, w, 1, \dots, 2n) \\ = \langle -g \partial_z \varphi(\bar{z}) \partial_w \varphi(w) \sigma(1) \cdots \sigma(2n) \rangle / \langle \sigma(1) \cdots \sigma(2n) \rangle \tag{4.39} \end{aligned}$$

one has also

$$H = \sum_{i,j=1}^n \mu^{(ij)}(1, \dots, 2n) \bar{\Omega}_1^{(i)}(\bar{z}) \Omega_1^{(j)}(w) \tag{4.40}$$

The constants λ, μ are then determined by the global monodromy requirements (4.9). Here one has to integrate over $2n$ contours, which separate into the $2n-2$ one considered in Ref. 18 and in Section 2 for calculations in the plane plus the two generators ω, ω' (an example for $n=2$ is given in Fig. 9). In the following we use the index $\alpha = 1, \dots, 2n$ for designating all these contours ($\mathcal{C}_{2n-1} = \omega, \mathcal{C}_{2n} = \omega'$), and we introduce the matrix

$$\Pi_1^{(\alpha,i)} = \int_{\mathcal{C}_\alpha} \Omega_1^{(i)}(z) dz, \quad \Pi_1^{(\alpha,i+n)} = \int_{\mathcal{C}_\alpha} \bar{\Omega}_1^{(i)}(\bar{z}) d\bar{z} \quad (4.41)$$

Then, one finds, after long, but straightforward algebra,

$$\begin{aligned} &\langle \sigma(1) \cdots \sigma(2n) \rangle_{qu} \\ &= \text{fct}(\tau) \prod_{\substack{ij=1 \\ i < j}}^{2n} \left| \frac{\theta_1'(0)}{\theta_1(z_{ij})} \right|^{1/4} \\ &\quad \times \prod_{\substack{k,l=1 \\ k < l}}^n |\theta_1(z_{kl})| \left| \theta_1 \left(\sum_{i=1}^n \frac{z_i - z_{i+n}}{2} \right) \right|^{n-1} \frac{1}{|\det \Pi_1|^{1/2}} \quad (4.42) \end{aligned}$$

where $\det \Pi_1$ is the determinant of the matrix (4.41). This result must in fact be independent of the choice of basis in (4.37).

We consider now the classical part of the correlator. In a soliton sector, φ_{cl} presents the usual discontinuities of $2\pi m, 2\pi m'$ along ω, ω' plus others also multiples of 2π along the $2n-2$ additional contours (Section 2). One has, analogous to (4.20),

$$\partial_z \varphi_{cl} = \sum_{i=1}^n A^{(i)} \Omega_1^{(i)}(z) \quad (4.43)$$

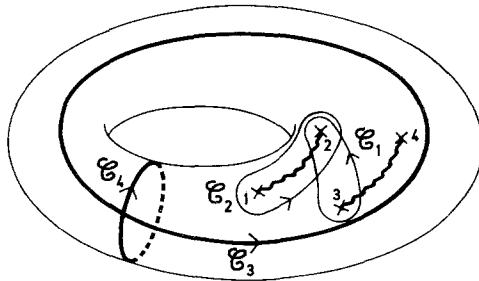


Fig. 9. The set of four contours used in the calculation of $\langle \sigma(1) \sigma(2) \sigma(3) \sigma(4) \rangle$.

subject to

$$\int_{\mathcal{C}_x} \partial_z \varphi_{cl} dz + \partial_{\bar{z}} \varphi_{cl} d\bar{z} = 2\pi m_\alpha \tag{4.44}$$

which is solved by $(A, \bar{A}) = \Pi_1^{-1}(m)$. The classical action thus reads

$$\begin{aligned} \mathcal{A}_{cl} &= 4\pi g \sum_{i,j=1}^n A^{(i)} \bar{A}^{(j)} \int_T \Omega_1^{(i)} \bar{\Omega}_1^{(j)} d^2x \\ &= 4\pi g \sum_{\alpha\beta=1}^m m_\alpha m_\beta \sum_{i,j=1}^n (\Pi_1^{-1})^{(i\alpha)} (\Pi_1^{-1})^{(j\beta)} \int_T \Omega_1^{(i)} \bar{\Omega}_1^{(j)} d^2x \end{aligned} \tag{4.45}$$

Finally one gets results in the twisted sectors by translating z_1 by the appropriate quantity, and the undetermined function in (4.42) can be obtained in principle in a recurrent way by short-distance expansions. Summing over $\{m_\alpha\} \in \mathbb{Z}$ gives the $2n$ spin function $\langle S_1 \cdots S_{2n} \rangle$. As explained in Section 2, one also can get mixed correlators involving S and t spins by adding $(-1)^{m_\alpha}$ factors for certain contours, exactly as in the plane.^(17,18) The corresponding expressions are, however, rather complicated.

5. TWIST CORRELATORS: RELATION TO THE APPROACH USING COVERING RIEMANN SURFACES

Twist correlators also have been calculated recently by a different approach, using covering Riemann surfaces. We discuss the compatibility of the results, restricting ourselves for simplicity to the two-point function. In what follows it will be useful to change slightly the notation for theta functions, introducing

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}} \exp[i\pi\tau(n+a)^2 + 2i\pi(n+a)(z+b)] \tag{5.1}$$

so that

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \theta_3, \quad \theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \theta_2, \quad \theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \theta_4, \quad \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \theta_1$$

We will also denote $\Omega_\nu, I_\nu, I'_\nu$ by $\Omega_{\alpha\beta}, I_{\alpha\beta}, I'_{\alpha\beta}$, where $\alpha\beta$ are the corresponding indices characterizing boundary conditions for φ .

The partition function with two twist insertions [the cut being contractible, as in (4.25)] was evaluated in Ref. 21 as (for a surface Σ of genus $G = 1$, i.e., a torus, which is the case of interest here)

$$\begin{aligned}
 Z_T[\sigma(1)\sigma(2)] &= \frac{Z_0}{2} (\tau_1)^{1/2} \left| \frac{\theta'_1(0, \tau)}{\theta_1(z_{12}, \tau)} \right|^{1/4} \\
 &\times \left| \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \right| \left| \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \right| \frac{Z_{cl}}{(\text{Im } \Pi)^{1/2}} \\
 &\times \left| \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \Pi) \right|^{-2} \tag{5.2}
 \end{aligned}$$

Z_0 is the partition function of the periodic free field (2.13). In the general case Π is the period matrix of Prym differentials W_i of the surface Σ with two branch points. These are differentials holomorphic everywhere except at the two points, where they behave as $\sim z^{-1/2}$, and their number is known⁽²¹⁾ to be the genus G of Σ . In our case $G = 1$, $W = \Omega_{00}(z) dz/I_{00}$ and Π is a number, $\Pi = I'_{00}/I_{00}$. On the other hand, Z_{cl} is given by

$$Z_{cl} = \sum_{mm' \in \mathbb{Z}} \exp \left[-\pi g \frac{|m' - m\Pi|^2}{\text{Im } \Pi} \right] \tag{5.3}$$

We thus see that (5.2) reproduces the $\nu = 1$ term in (4.28), provided

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) / \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(0, \frac{I'_{00}}{I_{00}} \right) = \frac{1}{I_{00}} \tag{5.4}$$

From (5.2) we deduce the partition functions where the twist line wraps around the torus by translating z_{12} of the appropriate quantity, and thus, by comparing with (4.28), relations analogous to (5.4) follow

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) / \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(0, \frac{I'_{\alpha\beta}}{I_{\alpha\beta}} \right) = \frac{1}{I_{\alpha\beta}} \tag{5.5}$$

One can check (5.4), (5.5) at first order in z_{12} , but I have not been able to demonstrate these formulas by a more direct approach, nor find them in the mathematical literature. If one accepts them nevertheless, one can recover some interesting results.

It can first be established that

$$\begin{aligned}
 &\theta \begin{bmatrix} \alpha + \gamma \\ \beta + \delta \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \theta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0, \tau) / \theta^2 \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \left(0, \frac{I'_{\alpha\beta}}{I_{\alpha\beta}} \right) \\
 &\text{is independent of } \gamma, \delta \in \{0, \frac{1}{2}\} \tag{5.6}
 \end{aligned}$$

as a limiting case of Schottky's relations⁽²⁹⁾ after pinching a cycle of the covering surface.⁽²¹⁾ Suppose now $g = 2$. Then, by identity (4.19) of Ref. 14 we have

$$\begin{aligned} & \frac{1}{[\text{Im}(I'_{\alpha\beta}/I_{\alpha\beta})]^{1/2}} \sum_{mm' \in \mathbb{Z}} \exp \left[-2\pi \frac{|m' - mI'_{\alpha\beta}/I_{\alpha\beta}|^2}{\text{Im}(I'_{\alpha\beta}/I_{\alpha\beta})} \right] \\ &= \frac{1}{2\sqrt{2}} \sum_{\gamma, \delta} \left| \theta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \left(0, \frac{I'_{\alpha\beta}}{I_{\alpha\beta}} \right) \right|^2 \end{aligned} \tag{5.7}$$

Using (5.5), (5.6), we get

$$\langle \sigma(1) \sigma(2) \rangle = \frac{1}{4Z^{\text{AT}} |\eta|^2} \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \sum_{\alpha\beta\gamma\delta} \left| \theta \begin{bmatrix} \alpha + \gamma \\ \beta + \delta \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \right| \left| \theta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0, \tau) \right| \tag{5.8}$$

i.e., since $Z^{\text{AT}} = (Z^1)^2$, (2.22),

$$\langle \sigma(1) \sigma(2) \rangle (g = 2) = \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \sum_{\alpha\beta} \left| \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \right| / \sum_{\alpha\beta} \left| \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \tau) \right| \tag{5.9}$$

reproducing indeed $\langle S(1) S(2) \rangle^{(1)}$, (4.34).

On the other hand, we have also doubling formulas at our disposal, giving⁽²⁹⁾

$$\begin{aligned} & \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{z_{12}}{2}, \tau \right) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) / \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(0, \frac{I'_{\alpha\beta}}{I_{\alpha\beta}} \right) \\ &= \theta^2 \begin{bmatrix} \gamma + \alpha/2 \\ \beta \end{bmatrix} \left(\frac{z_{12}}{2}, 2\tau \right) / \theta^2 \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \left(0, \frac{2I'_{\alpha\beta}}{I_{\alpha\beta}} \right) \end{aligned} \tag{5.10}$$

For $g = 1$ one can prove with identity (4.19) of Ref. 14 that

$$\begin{aligned} & \frac{1}{[\text{Im}(I'_{\alpha\beta}/I_{\alpha\beta})]^{1/2}} \sum_{mm' \in \mathbb{Z}} \exp \left[-\pi \frac{|m' - mI'_{\alpha\beta}/I_{\alpha\beta}|^2}{\text{Im}(I'_{\alpha\beta}/I_{\alpha\beta})} \right] \\ &= \sum_{\gamma} \left| \theta \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \left(0, \frac{2I'_{\alpha\beta}}{I_{\alpha\beta}} \right) \right|^2 \end{aligned} \tag{5.11}$$

Using (5.10), (5.11), we get

$$\langle \sigma(1) \sigma(2) \rangle (g = 1) = \frac{1}{2Z^{\text{AT}} |\eta|^2} \left| \frac{\theta'_1(0)}{\theta_1(z_{12})} \right|^{1/4} \sum_{\alpha\beta\gamma} \left| \theta \begin{bmatrix} \gamma + \alpha/2 \\ \beta \end{bmatrix} \left(\frac{z_{12}}{2}, 2\tau \right) \right|^2 \tag{5.12}$$

which can indeed be identified with (4.34bis), using same method as in Ref. 14.

6. CONCLUSION

I have shown how the correlation functions of the Ashkin–Teller model on a torus can be obtained from its free field mapping. I have mainly discussed the case of electromagnetic or twist operators, which imply different interesting technical approaches and have various applications. One could as well consider mixed correlators, which are obtained simply by combining the steps of Sections 3 and 4. I have discussed the compatibility of my formulas for twist correlation functions with those derived recently⁽²⁹⁾ by a different approach using covering Riemann surfaces, and shown that it implies interesting identities.

Note Added. At the final stage of this work I received a preprint by J. Attick, L. Dixon, A. Griffin, and D. Nemeschansky where results of the same nature as those of Section 4 are obtained in the context of string theory on Z_N orbifolds.

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